7.2 Lesson

Core Vocabulary

rational function, p. 366

Previous
domain
range
asymptote
long division

What You Will Learn

- Graph simple rational functions.
- Translate simple rational functions.
- Graph other rational functions.

Graphing Simple Rational Functions

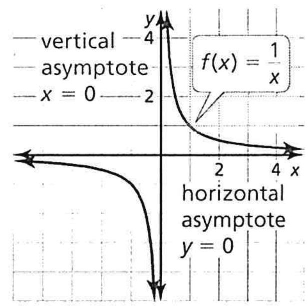
A **rational function** has the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. The inverse variation function $f(x) = \frac{a}{x}$ is a rational function. The graph of this function when $a = 1$ is shown below.

Core Concept

Parent Function for Simple Rational Functions

The graph of the parent function $f(x) = \frac{1}{x}$ is a *hyperbola*, which consists of two symmetrical parts called branches. The domain and range are all nonzero real numbers.

Any function of the form $g(x) = \frac{a}{x}$ $(a \neq 0)$ has the same asymptotes, domain, and range as the function $f(x) = \frac{1}{x}$.



STUDY TIP

Notice that $\frac{1}{x} \to 0$ as $x \to \infty$ and as $x \to -\infty$. This explains why $y = 0$ is a horizontal asymptote of the graph of $f(x) = \frac{1}{x}$. You can also analyze y-values as x approaches 0 to see why $x = 0$ is a vertical asymptote.

EXAMPLE 1 Graphing a Rational Function of the Form $y = \frac{a}{x}$

Graph $g(x) = \frac{4}{x}$. Compare the graph with the graph of $f(x) = \frac{1}{x}$.

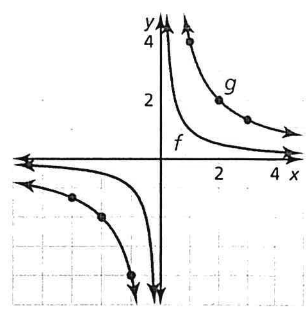
SOLUTION

Step 1 The function is of the form $g(x) = \frac{a}{x}$, so the asymptotes are $x = 0$ and $y = 0$. Draw the asymptotes.

Step 2 Make a table of values and plot the points. Include both positive and negative values of x.

x	-3	-2	-1	1	2	3
y	$-\frac{4}{3}$	-2	-4	4	2	$\frac{4}{3}$

Step 3 Draw the two branches of the hyperbola so that they pass through the plotted points and approach the asymptotes.



▶ The graph of g lies farther from the axes than the graph of f. Both graphs lie in the first and third quadrants and have the same asymptotes, domain, and range.

LOOKING FOR STRUCTURE

Because the function is of the form $g(x) = a \cdot f(x)$, where $a = 4$, the graph of g is a vertical stretch by a factor of 4 of the graph of f.

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1. Graph $g(x) = \frac{-6}{x}$. Compare the graph with the graph of $f(x) = \frac{1}{x}$.

Translating Simple Rational Functions

Core Concept

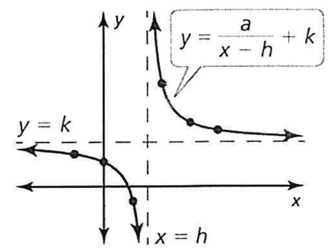
Graphing Translations of Simple Rational Functions

To graph a rational function of the form $y = \dfrac{a}{x - h} + k$, follow these steps:

Step 1 Draw the asymptotes $x = h$ and $y = k$.

Step 2 Plot points to the left and to the right of the vertical asymptote.

Step 3 Draw the two branches of the hyperbola so that they pass through the plotted points and approach the asymptotes.



EXAMPLE 2 Graphing a Translation of a Rational Function

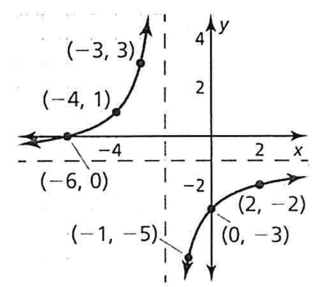
Graph $g(x) = \dfrac{-4}{x + 2} - 1$. State the domain and range.

SOLUTION

Step 1 Draw the asymptotes $x = -2$ and $y = -1$.

Step 2 Plot points to the left of the vertical asymptote, such as $(-3, 3)$, $(-4, 1)$, and $(-6, 0)$. Plot points to the right of the vertical asymptote, such as $(-1, -5)$, $(0, -3)$, and $(2, -2)$.

Step 3 Draw the two branches of the hyperbola so that they pass through the plotted points and approach the asymptotes.



▶ The domain is all real numbers except -2 and the range is all real numbers except -1.

LOOKING FOR STRUCTURE

Let $f(x) = \dfrac{-4}{x}$. Notice that g is of the form $g(x) = f(x - h) + k$, where $h = -2$ and $k = -1$. So, the graph of g is a translation 2 units left and 1 unit down of the graph of f.

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Graph the function. State the domain and range.

2. $y = \dfrac{3}{x} - 2$ **3.** $y = \dfrac{-1}{x + 4}$ **4.** $y = \dfrac{1}{x - 1} + 5$

Graphing Other Rational Functions

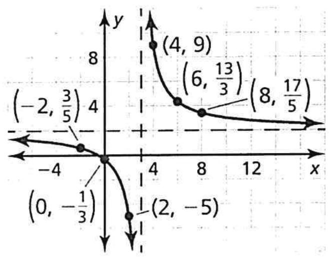
All rational functions of the form $y = \dfrac{ax + b}{cx + d}$ also have graphs that are hyperbolas.

- The vertical asymptote of the graph is the line $x = -\dfrac{d}{c}$ because the function is undefined when the denominator $cx + d$ is zero.
- The horizontal asymptote is the line $y = \dfrac{a}{c}$.

EXAMPLE 3 Graphing a Rational Function of the Form $y = \frac{ax + b}{cx + d}$

Graph $f(x) = \frac{2x + 1}{x - 3}$. State the domain and range.

SOLUTION



Step 1 Draw the asymptotes. Solve $x - 3 = 0$ for x to find the vertical asymptote $x = 3$. The horizontal asymptote is the line $y = \frac{a}{c} = \frac{2}{1} = 2$.

Step 2 Plot points to the left of the vertical asymptote, such as $(2, -5)$, $\left(0, -\frac{1}{3}\right)$, and $\left(-2, \frac{3}{5}\right)$. Plot points to the right of the vertical asymptote, such as $(4, 9)$, $\left(6, \frac{13}{3}\right)$, and $\left(8, \frac{17}{5}\right)$.

Step 3 Draw the two branches of the hyperbola so that they pass through the plotted points and approach the asymptotes.

▶ The domain is all real numbers except 3 and the range is all real numbers except 2.

Rewriting a rational function may reveal properties of the function and its graph. For example, rewriting a rational function in the form $y = \frac{a}{x - h} + k$ reveals that it is a translation of $y = \frac{a}{x}$ with vertical asymptote $x = h$ and horizontal asymptote $y = k$.

EXAMPLE 4 Rewriting and Graphing a Rational Function

Rewrite $g(x) = \frac{3x + 5}{x + 1}$ in the form $g(x) = \frac{a}{x - h} + k$. Graph the function. Describe the graph of g as a transformation of the graph of $f(x) = \frac{a}{x}$.

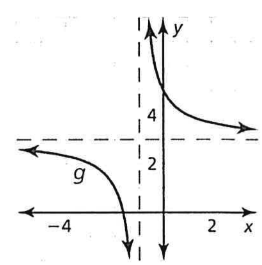
ANOTHER WAY

You will use a different method to rewrite g in Example 5 of Lesson 7.4.

SOLUTION

Rewrite the function by using long division:

$$\begin{array}{r} 3 \\ x + 1 \,\overline{)\, 3x + 5} \\ \underline{3x + 3} \\ 2 \end{array}$$



▶ The rewritten function is $g(x) = \frac{2}{x + 1} + 3$. The graph of g is a translation 1 unit left and 3 units up of the graph of $f(x) = \frac{2}{x}$.

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Graph the function. State the domain and range.

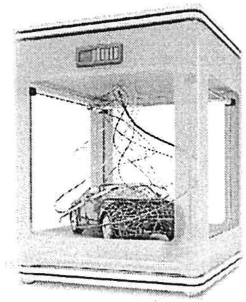
5. $f(x) = \frac{x - 1}{x + 3}$ **6.** $f(x) = \frac{2x + 1}{4x - 2}$ **7.** $f(x) = \frac{-3x + 2}{-x - 1}$

8. Rewrite $g(x) = \frac{2x + 3}{x + 1}$ in the form $g(x) = \frac{a}{x - h} + k$. Graph the function. Describe the graph of g as a transformation of the graph of $f(x) = \frac{a}{x}$.

EXAMPLE 5 Modeling with Mathematics

A 3-D printer builds up layers of materials to make three-dimensional models. Each deposited layer bonds to the layer below it. A company decides to make small display models of engine components using a 3-D printer. The printer costs $1000. The material for each model costs $50.

- Estimate how many models must be printed for the average cost per model to fall to $90.
- What happens to the average cost as more models are printed?



SOLUTION

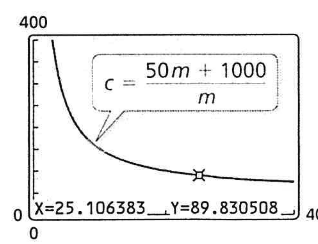
1. **Understand the Problem** You are given the cost of a printer and the cost to create a model using the printer. You are asked to find the number of models for which the average cost falls to $90.

2. **Make a Plan** Write an equation that represents the average cost. Use a graphing calculator to estimate the number of models for which the average cost is about $90. Then analyze the horizontal asymptote of the graph to determine what happens to the average cost as more models are printed.

3. **Solve the Problem** Let c be the average cost (in dollars) and m be the number of models printed.

$$c = \frac{\text{(Unit cost)(Number printed)} + \text{(Cost of printer)}}{\text{Number printed}} = \frac{50m + 1000}{m}$$

Use a graphing calculator to graph the function.



▶ Using the *trace* feature, the average cost falls to $90 per model after about 25 models are printed. Because the horizontal asymptote is $c = 50$, the average cost approaches $50 as more models are printed.

USING A GRAPHING CALCULATOR

Because the number of models and average cost cannot be negative, choose a viewing window in the first quadrant.

4. **Look Back** Use a graphing calculator to create tables of values for large values of m. The tables show that the average cost approaches $50 as more models are printed.

X	Y1
0	ERROR
50	70
100	60
150	56.667
200	55
250	54
300	53.333

X=0

X	Y1
0	ERROR
10000	50.1
20000	50.05
30000	50.033
40000	50.025
50000	50.02
60000	50.017

X=0

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9. **WHAT IF?** How do the answers in Example 5 change when the cost of the 3-D printer is $800?