7.3 Lesson

Core Vocabulary

rational expression, *p. 376*
simplified form of a rational expression, *p. 376*

Previous
fractions
polynomials
domain
equivalent expressions
reciprocal

What You Will Learn

- Simplify rational expressions.
- Multiply rational expressions.
- Divide rational expressions.

Simplifying Rational Expressions

A **rational expression** is a fraction whose numerator and denominator are nonzero polynomials. The *domain* of a rational expression excludes values that make the denominator zero. A rational expression is in **simplified form** when its numerator and denominator have no common factors (other than ± 1).

Core Concept

Simplifying Rational Expressions

Let a, b, and c be expressions with $b \neq 0$ and $c \neq 0$.

Property $\frac{a\cancel{c}}{b\cancel{c}} = \frac{a}{b}$ Divide out common factor c.

Examples $\frac{15}{65} = \frac{3 \cdot \cancel{5}}{13 \cdot \cancel{5}} = \frac{3}{13}$ Divide out common factor 5.

$\frac{4\cancel{(x+3)}}{(x+3)\cancel{(x+3)}} = \frac{4}{x+3}$ Divide out common factor $x + 3$.

STUDY TIP

Notice that you can divide out common factors in the second expression at the right. You cannot, however, divide out like terms in the third expression.

Simplifying a rational expression usually requires two steps. First, factor the numerator and denominator. Then, divide out any factors that are common to both the numerator and denominator. Here is an example:

$$\frac{x^2 + 7x}{x^2} = \frac{x(x+7)}{x \cdot x} = \frac{x+7}{x}$$

EXAMPLE 1 Simplifying a Rational Expression

Simplify $\frac{x^2 - 4x - 12}{x^2 - 4}$.

SOLUTION

$\frac{x^2 - 4x - 12}{x^2 - 4} = \frac{(x+2)(x-6)}{(x+2)(x-2)}$ Factor numerator and denominator.

$= \frac{\cancel{(x+2)}(x-6)}{\cancel{(x+2)}(x-2)}$ Divide out common factor.

$= \frac{x-6}{x-2}, \quad x \neq -2$ Simplified form

COMMON ERROR

Do not divide out variable terms that are not factors.

$$\frac{x-6}{x-2} \neq \frac{-6}{-2}$$

The original expression is undefined when $x = -2$. To make the original and simplified expressions equivalent, restrict the domain of the simplified expression by excluding $x = -2$. Both expressions are undefined when $x = 2$, so it is not necessary to list it.

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Simplify the rational expression, if possible.

1. $\frac{2(x+1)}{(x+1)(x+3)}$ **2.** $\frac{x+4}{x^2-16}$ **3.** $\frac{4}{x(x+2)}$ **4.** $\frac{x^2-2x-3}{x^2-x-6}$

Multiplying Rational Expressions

The rule for multiplying rational expressions is the same as the rule for multiplying numerical fractions: multiply numerators, multiply denominators, and write the new fraction in simplified form. Similarly to rational numbers, rational expressions are closed under multiplication.

Core Concept

Multiplying Rational Expressions

Let a, b, c, and d be expressions with $b \neq 0$ and $d \neq 0$.

Property $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ Simplify $\frac{ac}{bd}$ if possible.

Example $\frac{5x^2}{2xy^2} \cdot \frac{6xy^3}{10y} = \frac{30x^3y^3}{20xy^3} = \frac{\cancel{10} \cdot 3 \cdot \cancel{x} \cdot x^2 \cdot \cancel{y^3}}{\cancel{10} \cdot 2 \cdot \cancel{x} \cdot \cancel{y^3}} = \frac{3x^2}{2}, \quad x \neq 0, y \neq 0$

ANOTHER WAY

In Example 2, you can first simplify each rational expression, then multiply, and finally simplify the result.

$$\frac{8x^3y}{2xy^2} \cdot \frac{7x^4y^3}{4y}$$

$$= \frac{4x^2}{y} \cdot \frac{7x^4y^2}{4}$$

$$= \frac{\cancel{4} \cdot 7 \cdot x^6 \cdot \cancel{y} \cdot y}{\cancel{4} \cdot \cancel{y}}$$

$$= 7x^6y, \quad x \neq 0, y \neq 0$$

EXAMPLE 2 Multiplying Rational Expressions

Find the product $\frac{8x^3y}{2xy^2} \cdot \frac{7x^4y^3}{4y}$.

SOLUTION

$\frac{8x^3y}{2xy^2} \cdot \frac{7x^4y^3}{4y} = \frac{56x^7y^4}{8xy^3}$ Multiply numerators and denominators.

$= \frac{\cancel{8} \cdot 7 \cdot \cancel{x} \cdot x^6 \cdot \cancel{y^3} \cdot y}{\cancel{8} \cdot \cancel{x} \cdot \cancel{y^3}}$ Factor and divide out common factors.

$= 7x^6y, \quad x \neq 0, y \neq 0$ Simplified form

EXAMPLE 3 Multiplying Rational Expressions

Find the product $\frac{3x - 3x^2}{x^2 + 4x - 5} \cdot \frac{x^2 + x - 20}{3x}$.

SOLUTION

$\frac{3x - 3x^2}{x^2 + 4x - 5} \cdot \frac{x^2 + x - 20}{3x} = \frac{3x(1 - x)}{(x - 1)(x + 5)} \cdot \frac{(x + 5)(x - 4)}{3x}$ Factor numerators and denominators.

$= \frac{3x(1 - x)(x + 5)(x - 4)}{(x - 1)(x + 5)(3x)}$ Multiply numerators and denominators.

$= \frac{3x(-1)(x - 1)(x + 5)(x - 4)}{(x - 1)(x + 5)(3x)}$ Rewrite $1 - x$ as $(-1)(x - 1)$.

$= \frac{\cancel{3x}(-1)\cancel{(x - 1)}\cancel{(x + 5)}(x - 4)}{\cancel{(x - 1)}\cancel{(x + 5)}\cancel{(3x)}}$ Divide out common factors.

$= -x + 4, \quad x \neq -5, x \neq 0, x \neq 1$ Simplified form

Check

X	Y1	Y2
−5	ERROR	9
−4	8	8
−3	7	7
−2	6	6
−1	5	5
0	ERROR	4
1	ERROR	3

X=−4

Check the simplified expression. Enter the original expression as y_1 and the simplified expression as y_2 in a graphing calculator. Then use the *table* feature to compare the values of the two expressions. The values of y_1 and y_2 are the same, except when $x = -5$, $x = 0$, and $x = 1$. So, when these values are excluded from the domain of the simplified expression, it is equivalent to the original expression.

EXAMPLE 4 Multiplying a Rational Expression by a Polynomial

STUDY TIP

Notice that $x^2 + 3x + 9$ does not equal zero for any real value of x. So, no values must be excluded from the domain to make the simplified form equivalent to the original.

Find the product $\frac{x+2}{x^3-27} \cdot (x^2+3x+9)$.

SOLUTION

$$\frac{x+2}{x^3-27} \cdot (x^2+3x+9) = \frac{x+2}{x^3-27} \cdot \frac{x^2+3x+9}{1}$$ Write polynomial as a rational expression.

$$= \frac{(x+2)(x^2+3x+9)}{(x-3)(x^2+3x+9)}$$ Multiply. Factor denominator.

$$= \frac{(x+2)\cancel{(x^2+3x+9)}}{(x-3)\cancel{(x^2+3x+9)}}$$ Divide out common factors.

$$= \frac{x+2}{x-3}$$ Simplified form

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Find the product.

5. $\frac{3x^5y^2}{8xy} \cdot \frac{6xy^2}{9x^3y}$

6. $\frac{2x^2-10x}{x^2-25} \cdot \frac{x+3}{2x^2}$

7. $\frac{x+5}{x^3-1} \cdot (x^2+x+1)$

Dividing Rational Expressions

To divide one rational expression by another, multiply the first rational expression by the reciprocal of the second rational expression. Rational expressions are closed under nonzero division.

Core Concept

Dividing Rational Expressions

Let a, b, c, and d be expressions with $b \neq 0$, $c \neq 0$, and $d \neq 0$.

Property $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$ Simplify $\frac{ad}{bc}$ if possible.

Example $\frac{7}{x+1} \div \frac{x+2}{2x-3} = \frac{7}{x+1} \cdot \frac{2x-3}{x+2} = \frac{7(2x-3)}{(x+1)(x+2)}, x \neq \frac{3}{2}$

EXAMPLE 5 Dividing Rational Expressions

Find the quotient $\frac{7x}{2x-10} \div \frac{x^2-6x}{x^2-11x+30}$.

SOLUTION

$$\frac{7x}{2x-10} \div \frac{x^2-6x}{x^2-11x+30} = \frac{7x}{2x-10} \cdot \frac{x^2-11x+30}{x^2-6x}$$ Multiply by reciprocal.

$$= \frac{7x}{2(x-5)} \cdot \frac{(x-5)(x-6)}{x(x-6)}$$ Factor.

$$= \frac{7\cancel{x}\cancel{(x-5)}\cancel{(x-6)}}{2\cancel{(x-5)}\cancel{(x)}\cancel{(x-6)}}$$ Multiply. Divide out common factors.

$$= \frac{7}{2}, \quad x \neq 0, x \neq 5, x \neq 6$$ Simplified form

EXAMPLE 6 Dividing a Rational Expression by a Polynomial

Find the quotient $\frac{6x^2 + x - 15}{4x^2} \div (3x^2 + 5x)$.

SOLUTION

$$\frac{6x^2 + x - 15}{4x^2} \div (3x^2 + 5x) = \frac{6x^2 + x - 15}{4x^2} \cdot \frac{1}{3x^2 + 5x} \qquad \text{Multiply by reciprocal.}$$

$$= \frac{(3x + 5)(2x - 3)}{4x^2} \cdot \frac{1}{x(3x + 5)} \qquad \text{Factor.}$$

$$= \frac{\cancel{(3x + 5)}(2x - 3)}{4x^2(x)\cancel{(3x + 5)}} \qquad \text{Divide out common factors.}$$

$$= \frac{2x - 3}{4x^3}, \quad x \neq -\frac{5}{3} \qquad \text{Simplified form}$$

EXAMPLE 7 Solving a Real-Life Problem

The total annual amount I (in millions of dollars) of personal income earned in Alabama and its annual population P (in millions) can be modeled by

$$I = \frac{6922t + 106{,}947}{0.0063t + 1}$$

and

$$P = 0.0343t + 4.432$$

where t represents the year, with $t = 1$ corresponding to 2001. Find a model M for the annual per capita income. (Per capita means per person.) Estimate the per capita income in 2010. (Assume $t > 0$.)

SOLUTION

To find a model M for the annual per capita income, divide the total amount I by the population P.

$$M = \frac{6922t + 106{,}947}{0.0063t + 1} \div (0.0343t + 4.432) \qquad \text{Divide } I \text{ by } P.$$

$$= \frac{6922t + 106{,}947}{0.0063t + 1} \cdot \frac{1}{0.0343t + 4.432} \qquad \text{Multiply by reciprocal.}$$

$$= \frac{6922t + 106{,}947}{(0.0063t + 1)(0.0343t + 4.432)} \qquad \text{Multiply.}$$

To estimate Alabama's per capita income in 2010, let $t = 10$ in the model.

$$M = \frac{6922 \cdot 10 + 106{,}947}{(0.0063 \cdot 10 + 1)(0.0343 \cdot 10 + 4.432)} \qquad \text{Substitute 10 for } t.$$

$$\approx 34{,}707 \qquad \text{Use a calculator.}$$

▶ In 2010, the per capita income in Alabama was about \$34,707.

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Find the quotient.

8. $\frac{4x}{5x - 20} \div \frac{x^2 - 2x}{x^2 - 6x + 8}$

9. $\frac{2x^2 + 3x - 5}{6x} \div (2x^2 + 5x)$