Mixed Practice #1

1. $\lim_{x\to 0} \frac{x-\sin x}{\tan x}\ \frac{0}{0} \Rightarrow \lim_{x\to 0} \frac{1-\cos x}{\sec^2 x} = \lim_{x\to 0} \cancel{\cos^2 x}(1-\cancel{\cos x})$

$= 1(0) = 0$

2. $\lim_{x\to 0^+} x^{\sin 3x}$

$y = x^{\sin 3x}$

$\ln y = \sin 3x \cdot \ln x$

$\lim_{x\to 0} \ln y = \lim_{x\to 0} \sin 3x \cdot \ln x$

$\ln\left(\lim_{x\to 0} y\right) = \lim_{x\to 0} \frac{\ln x}{\csc 3x}\ \frac{-\infty}{\infty}$

$= \lim_{x\to 0} \frac{\frac{1}{x}}{-3\csc 3x \cot 3x}$

$= \lim_{x\to 0} \frac{\sin 3x \tan 3x}{-3x} = \lim_{x\to 0} \frac{\sin^2 3x}{-3\cos 3x \cdot x}\ \frac{0}{0}$

$= \lim_{x\to 0} \frac{2\sin 3x \cos 3x \cdot \cancel{3}}{-\cancel{3}(\cos 3x + -3x \sin 3x)}$

$= \lim_{x\to 0} \frac{-2\cancel{\sin 3x}\,\cancel{\cos 3x}}{\cancel{\cos 3x} - 3\cancel{x}\,\cancel{\sin 3x}} = \frac{-2(0)1}{1-0\cdot 0}$

$= \frac{0}{1} = 0$

$\cancel{e^{\ln}} \lim_{x\to 0} y = e^{0}$

$\lim_{x\to 0} y = e^0 = 1$

So, $\lim_{x\to 0^+} x^{\sin 3x} = 1$

3. $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{x^2}\right)$ $\infty - \infty$ $= \lim_{x\to 0}\left(\frac{x-1}{x^2}\right)\ \frac{0}{0} \Rightarrow \lim_{x\to 0}\left(\frac{1}{2x}\right) - \frac{1}{0}$

undefined.

4. $$\lim_{x\to\infty} \frac{\sqrt{x^2+x}}{x} \;\frac{\infty}{\infty} \Rightarrow \lim_{x\to\infty} \frac{\frac{1}{2}(x^2+x)^{-\frac{1}{2}}(2x+1)}{1}$$

$$= \lim_{x\to\infty} \frac{2x+1}{2(x^2+x)^{1/2}} \;\frac{\infty}{\infty}$$ Not helpful!

$$\lim_{x\to\infty} \frac{\sqrt{x^2\left(1+\frac{1}{x}\right)}}{x} = \lim_{x\to\infty} \frac{\cancel{x}\sqrt{1+\frac{1}{x}}}{\cancel{x}} = \lim_{x\to\infty} \sqrt{1+\underbrace{\cancel{\tfrac{1}{x}}}_{0}} = 1$$

So, they grow at the same rate.

5. $$\lim_{x\to\infty} \frac{\ln x}{x-1} \;\frac{\infty}{\infty} \Rightarrow \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

So, $x-1$ grows faster than $\ln x$.

6. $$\lim_{x\to\infty} \frac{x^3}{\ln(\ln x)} \;\frac{\infty}{\infty} \Rightarrow \lim_{x\to\infty} \frac{3x^2}{\frac{1}{\ln x}\cdot\frac{1}{x}} = \lim_{x\to\infty} 3x^3 \cdot \ln x \to \infty$$

So, x^3 grows faster than $\ln(\ln x)$.

7. $$-\frac{1}{2}\int_0^1 -2\sqrt[5]{1-2x}\,dx \Rightarrow -\frac{1}{2}\int_0^1 (1-2x)^{1/5}(-2\,dx) = -\frac{1}{2}\cdot\frac{5}{6}(1-2x)^{6/5}\Big|_0^1$$

$u = 1-2x$
$du = -2\,dx$

$$= \frac{-5}{12}(1-2x)^{6/5}\Big|_0^1 = \frac{-5}{12}\Big(\underbrace{\cancel{(-1)^{6/5} - 1}}_{0}\Big) = 0$$

8. $$\int_0^{\pi/2} \sin^4 x \cos x\,dx = \int_0^{\pi/2} (\sin x)^4 \cos x\,dx = \frac{\sin^5 x}{5}\Big|_0^{\pi/2}$$

$u = \sin x$
$du = \cos x\,dx$

$$= \frac{\overset{1}{\cancel{\sin^5(\pi/2)}}}{5} - \frac{\overset{0}{\cancel{\sin^5(0)}}}{5} = \frac{1}{5}$$

9. $\int \frac{x-2}{x-1}\,dx = \int\left(1 + \frac{-1}{x-1}\right)dx = x - \ln|x-1| + C$

$$\begin{array}{r} 1 \\ x-1 \overline{)\,x-2} \\ \underline{-x+1} \\ -1 \end{array}$$

10. $\int_3^\infty \frac{1}{x}\,dx \Rightarrow \lim_{a\to\infty} \int_3^a \frac{1}{x}\,dx = \lim_{a\to\infty} \ln|x| \Big|_3^a$

$= \lim_{a\to\infty} \ln|a| - \ln|3| \Rightarrow \infty$ Diverges

11. $\int_0^\infty \frac{x}{4+x^2}\,dx = \lim_{a\to\infty} \frac{1}{2}\int_0^a \frac{2x}{4+x^2}\,dx = \lim_{a\to\infty} \frac{1}{2}\ln|4+x^2| \Big|_0^a$

$u = 4+x^2$
$du = 2x\,dx$

$= \lim_{a\to\infty} \frac{1}{2}|4+a^2| - \frac{1}{2}\ln|4| \to \infty$ Diverges

12. $\int_0^4 \frac{\ln\sqrt{x}}{\sqrt{x}}\,dx = \lim_{a\to 0} 2\int_a^4 \frac{\ln\sqrt{x}}{2\sqrt{x}} \cdot dx \Rightarrow \lim_{a\to 0} 2\int \ln u \cdot du$

$u = \sqrt{x}$
$du = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}\,dx$

$= \lim_{a\to 0} 2(u\ln u - u) \Rightarrow \lim_{a\to 0} 2(\sqrt{x}\ln(\sqrt{x}) - \sqrt{x})\Big|_a^4$

$= \lim_{a\to 0} 2\left[\sqrt{4}\cdot\ln\sqrt{4} - \sqrt{4} - (\sqrt{a}\ln\sqrt{a} - \sqrt{a})\right]$

$= 2(2\ln 2 - 2) - 2\lim_{a\to 0}(\overset{0\,\cdot\,\infty}{\sqrt{a}\ln\sqrt{a}} - \cancel{\sqrt{a}}^{\,0})$

$= 2(2\ln 2 - 2) - 2\lim_{a\to 0} \frac{\ln\sqrt{a}}{\frac{1}{\sqrt{a}}} = 4\ln 2 - 4 - 2\lim_{a\to 0} \frac{\frac{1}{\sqrt{a}}\cdot\frac{1}{2\sqrt{a}}}{-\frac{1}{2}a^{-3/2}}$

$= 4\ln 2 - 4 - 2\lim_{a\to 0} \frac{1}{2a}\cdot\frac{-2a^{3/2}}{1} = 4\ln 2 - 4 - \cancel{2\lim_{a\to 0} -a^{1/2}}^{\,0} = \boxed{4\ln 2 - 4}$